

# SECANT VARIETIES OF SEGRE-VERONESE EMBEDDINGS OF $(\mathbb{P}^1)^r$ .

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ABSTRACT. We use a double degeneration technique to calculate the dimension of the secant variety of any Segre-Veronese embedding of  $(\mathbb{P}^1)^r$ .

## INTRODUCTION

The problem of determining the dimension of linear systems through double points in general position on an algebraic variety  $X$  is a very hard one to be solved in general. Complete results are known for varieties of small dimension [BD10, CC02, Laf02, CGG07, VT05] and in any dimension just for  $\mathbb{P}^n$  by the Alexander-Hirschowitz theorem [AH95, Pos10].

Secant varieties of Segre-Veronese varieties are not well-understood, so far, and many efforts have been made, see for example [CGG05], [Bal06], [Abr08], [AB09], [AB11]. In this paper we determine the dimension of any linear system  $\mathcal{L}_{(d_1, \dots, d_r)}(2^n)$  of hypersurfaces of  $(\mathbb{P}^1)^r$  of multi-degree  $(d_1, \dots, d_r)$  through  $n$  double points in general position. Solving this problem is equivalent, via Terracini's Lemma, to calculate the dimension of the secant variety of any Segre-Veronese embedding  $(\mathbb{P}^1)^r \rightarrow \mathbb{P}^N$ , defined by the complete linear system  $|\mathcal{O}(d_1, \dots, d_r)|$  of  $(\mathbb{P}^1)^r$ . Our proof is based on a double induction, on the dimension  $r$  and on the degree  $d_1 + \dots + d_r$ . A basic step for our induction is represented by the fundamental paper [CGG11], where the authors show that, if all the  $d_i = 1$ , then  $\mathcal{L}_{(1, \dots, 1)}(2^n)$  has always but in one case ( $r = 4$ ) the expected dimension.

Our approach consists in degenerating  $(\mathbb{P}^1)^r$  to a union of two varieties both isomorphic to  $(\mathbb{P}^1)^r$  and simultaneously degenerating the linear system to a linear system obtained as fibered product of linear systems on the two components over the restricted system on their intersection. The limit linear system is somewhat easier than the original one, in particular this degeneration argument allows to use induction on the multi-degree and on  $r$ . This construction is a generalization of the technique introduced by Ciliberto and Miranda in [CM98] and [CM00] to study higher multiplicity interpolation problems in  $\mathbb{P}^2$  and recently generalized in [Pos10] to the higher dimensional case to study linear systems of  $d$ -hypersurfaces of  $\mathbb{P}^r$  with a general collection of nodes.

The paper is organized as follows. In Section 1 we provide the basic notation. Section 2 contains the statement of our main theorem together with its counterpart on secant varieties. We enter the double degeneration technique in Section 3, while

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Section 4 is devoted to apply this technique to compute the dimension of linear systems. In Section 5 we prove our theorem and finally we conclude in Section 6 by describing all the linear systems whose dimension is not the expected one.

## 1. NOTATION

Let  $\mathcal{L} := \mathcal{L}_{(d_1, \dots, d_r)}(2^n)$  be the linear system of multi-degree  $(d_1, \dots, d_r)$  hypersurfaces of  $(\mathbb{P}^1)^r$  which are singular at  $n$  points in general position. Its *virtual dimension* is defined to be

$$v(\mathcal{L}) := \prod_{i=1}^r (d_i + 1) - 1 - (r + 1)n,$$

i.e. the dimension of the linear system  $|\mathcal{O}(d_1, \dots, d_r)|$  of multi-degree  $(d_1, \dots, d_r)$  hypersurfaces of  $(\mathbb{P}^1)^r$  minus the number of conditions imposed by the double points. The dimension of  $\mathcal{L}$  cannot be less than  $-1$ , hence we define the *expected dimension* to be

$$e(\mathcal{L}) := \max\{v(\mathcal{L}), -1\}.$$

If the conditions imposed by the assigned points are not linearly independent, the dimension of  $\mathcal{L}$  is greater than the expected one: in that case we say that  $\mathcal{L}$  is *special*. Otherwise, if the dimension and the expected dimension of  $\mathcal{L}$  coincide, we say that  $\mathcal{L}$  is *non-special*.

We are interested in investigating if a given linear system  $\mathcal{L}$  is non-special. The dimension of  $\mathcal{L}$  is *upper-semicontinuous* in the position of the points in  $(\mathbb{P}^1)^n$  and it achieves its minimum value when they are in *general position*. Let  $Z$  be the zero-dimensional scheme of length  $(r + 1)n$  given by  $n$  double points in general position. With abuse of notation we will sometimes adopt the same symbol  $\mathcal{L}_{(d_1, \dots, d_r)}(2^n)$ , or simply  $\mathcal{L}$ , for denoting the linear system and the sheaf  $\mathcal{O}(d_1, \dots, d_r) \otimes \mathcal{I}_Z$ . With this in mind, consider the following restriction exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_{(d_1, \dots, d_r)} \longrightarrow \mathcal{L}_{(d_1, \dots, d_r)}|_Z.$$

Taking cohomology, being  $h^1((\mathbb{P}^1)^r, \mathcal{L}_{(d_1, \dots, d_r)}) = 0$  we get that  $\mathcal{L}$  is non-special if and only if

$$h^0((\mathbb{P}^1)^r, \mathcal{L}) \cdot h^1((\mathbb{P}^1)^r, \mathcal{L}) = 0.$$

## 2. THE CLASSIFICATION THEOREM

In this section we state our main theorem and recall its connection with the dimension of the secant varieties of the Segre-Veronese embeddings of  $(\mathbb{P}^1)^r$ .

**Theorem 2.1.** *The linear system  $\mathcal{L}_{(d_1, \dots, d_r)}(2^n)$  of  $(\mathbb{P}^1)^r$  is non-special except in the following cases.*

$r$	degrees	$n$	$v(\mathcal{L})$	$\dim(\mathcal{L})$
2	(2, 2a)	$2a + 1$	-1	0
3	(1, 1, 2a)	$2a + 1$	-1	0
	(2, 2, 2)	7	-2	0
4	(1, 1, 1, 1)	3	0	1

All the exception in Theorem 2.1 where previously known. Moreover a proof of the two dimensional case can be found in [Laf02, VT05]. The three dimensional case was treated in [BD10, CGG07]. The special system in dimension 4 has been found in [CGG05].

This theorem has an equivalent reformulation in terms of higher secant varieties of Segre-Veronese embeddings of products of  $\mathbb{P}^1$ 's. Let  $X$  be a projective variety of dimension  $r$  embedded in  $\mathbb{P}^N$ . The  $n$ -secant variety  $S_n(X)$  of  $X$  is defined to be the Zariski closure of the union of all the linear spans in  $\mathbb{P}^N$  of  $n$ -tuples of independent points of  $X$ . We have, counting parameters, that

$$\dim(S_n(X)) \leq \min\{nr + n - 1, N\}$$

The variety  $X$  is said to be  $n$ -defective if strict inequality holds; it is said to be non- $n$ -defective if equality holds.

Let  $N := \prod_{i=1}^r (d_i + 1) - 1$  and let  $\nu : (\mathbb{P}^1)^r \rightarrow \mathbb{P}^N$  be the Segre-Veronese embedding of multi-degree  $(d_1, \dots, d_r)$ . Denote by  $X$  the image of  $\nu$ .

**Theorem 2.2.** *Let  $X$  be defined as above. The  $n$ -secant variety of  $X$  is non-defective, with the same list of exceptions of Theorem 2.1.*

A hypersurface  $S$  of  $(\mathbb{P}^1)^r$  of multi-degree  $(d_1, \dots, d_r)$  corresponds via the Segre-Veronese embedding to a hyperplane section  $H$  of  $X$ . Moreover  $S$  has a double point at  $p$  if and only if  $H$  is tangent to  $X$  at  $\nu(p)$ . Now, fix  $p_1, \dots, p_n$  general points in  $(\mathbb{P}^1)^r$  and consider the linear system  $\mathcal{L}_{(d_1, \dots, d_r)}(2^n)$  of multi-degree  $(d_1, \dots, d_r)$  hypersurfaces singular at  $p_1, \dots, p_n$ . It corresponds to the linear system of hyperplanes  $H$  in  $\mathbb{P}^N$  tangent to  $X$  at  $\nu(p_1), \dots, \nu(p_n)$ . This linear system has as base locus the general tangent space to  $S_n(X)$ . The following classical result, known as Terracini Lemma, proves the equivalence between Theorem 2.1 and Theorem 2.2.

**Lemma 2.3** (Terracini's Lemma). *Let  $X \subseteq \mathbb{P}^N$  be an irreducible, non-degenerate, projective variety of dimension  $r$ . Let  $p_1, \dots, p_n$  be general points of  $X$ , with  $n \leq N + 1$ . Then*

$$T_{S_n(X), p} = \langle T_{X, p_1}, \dots, T_{X, p_n} \rangle,$$

where  $p \in \langle p_1, \dots, p_n \rangle$  is a general point in  $S_n(X)$ .

### 3. THE DEGENERATION TECHNIQUE

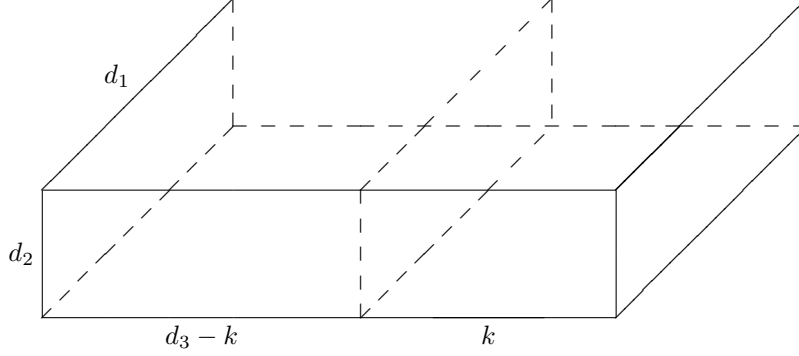
We begin by constructing a toric flat degeneration of the variety  $X$  into a union of two toric varieties  $X_1$  and  $X_2$  both isomorphic to  $X$ .

**3.1. A toric degeneration of  $(\mathbb{P}^1)^r$ .** Let  $P = P_{(d_1, \dots, d_r)}$  be the convex lattice polytope  $[0, d_1] \times \dots \times [0, d_r] \subseteq \mathbb{R}^r$ . Its integer points define the toric map which is the Segre-Veronese embedding  $\nu : (\mathbb{P}^1)^r \rightarrow \mathbb{P}^N$  given by the line bundle  $\mathcal{O}(d_1, \dots, d_r)$ . As before we will denote by  $X$  the image of  $\nu$ . Consider the function  $\phi : P \cap \mathbb{Z}^r \rightarrow \mathbb{Z}$  defined by

$$\phi(v) = \begin{cases} 0 & \text{if } v_r \leq k \\ v_r - k & \text{if } v_r > k. \end{cases}$$

It defines a regular subdivision of  $P$  in the following way. Consider the convex hull of the half lines  $\{(v, t) \in P \times \mathbb{R}_{\geq 0} : t \geq \phi(v)\}$ . This is an unbounded polyhedra with two lower faces. By projecting these faces on  $P$  we obtain the regular subdivision

$$\mathcal{T} = \{P_{(d_1, \dots, d_{r-1}, d_r - k)}, P_{(d_1, \dots, d_{r-1}, k)}\},$$

FIGURE 1. A regular subdivision of  $P_{(d_1, d_2, d_3)}$ 

with  $P_{(d_1, \dots, d_{r-1}, d_r - k)} \cup P_{(d_1, \dots, d_{r-1}, k)} = P$  and  $P_{(d_1, \dots, d_{r-1}, d_r - k)} \cap P_{(d_1, \dots, d_{r-1}, k)} = P_{(d_1, \dots, d_{r-1})}$ , where  $P_{(d_1, \dots, d_{r-1})} = [0, d_1] \times \dots \times [0, d_{r-1}]$ . We show the configuration of this toric degeneration in Figure 1 for  $r = 3$ . The regular subdivision defines a 1-dimensional embedded degeneration in the following way. Let  $\mathcal{X}$  be the toric subvariety of  $\mathbb{P}^N \times \mathbb{A}^1$  which is image of the toric morphism

$$(\mathbb{C}^*)^r \times \mathbb{C}^* \rightarrow \mathbb{P}^N \times \mathbb{A}^1 \quad (x, t) \mapsto (\{t^{\phi(v)} x^v : v \in P \cap \mathbb{Z}^r\}, t).$$

Then  $\mathcal{X}$  admits two morphisms, induced by the projections, onto  $\mathbb{P}^N$  and  $\mathbb{A}^1$ . Denote by  $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$  the second morphism. The fiber  $X_t$  of  $\pi$  over  $t$  is isomorphic to  $X$  if  $t$  is not null and to the union  $X_1 \cup X_2$  if  $t = 0$ . Both the  $X_i$ 's are isomorphic to  $(\mathbb{P}^1)^r$  and their intersection  $R$  is isomorphic to  $(\mathbb{P}^1)^{r-1}$ . More precisely,  $X_1$  is the Segre-Veronese embedding of  $(\mathbb{P}^1)^r$  defined by the complete linear system  $|\mathcal{O}(d_1, \dots, d_{r-1}, d_r - k)|$ , while  $X_2$  is the Segre-Veronese embedding of  $(\mathbb{P}^1)^r$  defined by  $|\mathcal{O}(d_1, \dots, d_{r-1}, k)|$ . The intersection  $R$  is the Segre-Veronese embedding of  $(\mathbb{P}^1)^{r-1}$  given by  $|\mathcal{O}(d_1, \dots, d_{r-1})|$ .

**3.2. The  $(k, n_2)$ -degeneration of  $\mathcal{L}$ .** A line bundle on  $X_0$  corresponds to two line bundles, respectively on  $X_1$  and on  $X_2$ , which agree on the intersection  $R$ . We consider the linear system  $\mathcal{L}_t := \mathcal{L}$  of multi-degree  $(d_1, \dots, d_r)$  hypersurfaces of  $X$  with  $n$  assigned general points  $p_{1,t}, \dots, p_{n,t}$  of multiplicity 2.

Fix a non-negative integer  $n_1 \leq n$  and specialize  $n_1$  points generically on  $X_1$  and the other  $n_2 = n - n_1$  points generically on  $X_2$ , i.e. take a flat family  $\{p_{1,t}, \dots, p_{n,t}\}_{t \in \mathbb{A}^1}$  such that  $p_{1,0}, \dots, p_{n_1,0} \in X_1$  and  $p_{n_1+1,0}, \dots, p_{n,0} \in X_2$ . The limiting linear system  $\mathcal{L}_0$  on  $X_0$  is formed by the divisors in the flat limit of the bundle  $\mathcal{O}(d_1, \dots, d_r)$  on the general fiber  $X_t$ , singular at  $p_{1,0}, \dots, p_{n,0}$ . Consider the following linear systems:

$$(3.1) \quad \begin{aligned} \mathcal{L}_1 &:= \mathcal{L}_{(d_1, \dots, d_{r-1}, d_r - k)}(2^{n_1}) & \mathcal{L}_2 &:= \mathcal{L}_{(d_1, \dots, d_{r-1}, k)}(2^{n_2}) \\ \hat{\mathcal{L}}_1 &:= \mathcal{L}_{(d_1, \dots, d_{r-1}, d_r - k - 1)}(2^{n_1}) & \hat{\mathcal{L}}_2 &:= \mathcal{L}_{(d_1, \dots, d_{r-1}, k - 1)}(2^{n_2}), \end{aligned}$$

where  $\mathcal{L}_i, \hat{\mathcal{L}}_i$  are defined on  $X_i$  and  $\hat{\mathcal{L}}_i$  is the kernel of the restriction of  $\mathcal{L}_i$  to  $R$ . This is given by the exact sequence:

$$0 \longrightarrow \hat{\mathcal{L}}_i \longrightarrow \mathcal{L}_i \longrightarrow \mathcal{L}_i|_R =: \mathcal{R}_i \longrightarrow 0$$

The kernel  $\hat{\mathcal{L}}_i$  consists of those divisors of  $\mathcal{L}_i$  which vanish identically on  $R$ , i.e. the divisors in  $\mathcal{L}_i$  containing  $R$  as component. An element of  $\mathcal{L}_0$  consists either of a divisor on  $X_1$  and a divisor on  $X_2$ , both satisfying the conditions imposed by the multiple points, which restrict to the same divisor on  $R$ , or it is a divisor corresponding to a section of the bundle which is identically zero on  $X_1$  (or on  $X_2$ ) and which gives a general divisor in  $\mathcal{L}_2$  (or in  $\mathcal{L}_1$  respectively) containing  $R$  as a component. We have, by upper-semicontinuity, that  $\dim(\mathcal{L}_0) \geq \dim(\mathcal{L})$ .

**Lemma 3.1.** *In the above notation, if  $\dim(\mathcal{L}_0) = e(\mathcal{L})$ , then the linear system  $\mathcal{L}$  has the expected dimension, i.e. it is non-special.*

We will say that  $\mathcal{L}_0$  is obtained from  $\mathcal{L}$  by a  $(k, n_2)$ -degeneration.

**3.3. The  $(k, n_2, \beta)$ -degeneration of  $\mathcal{L}$ .** Fix a non-negative integer  $\beta < \min(r, n_2)$ . Suppose that we have already performed a  $(k, n_2)$ -degeneration of  $\mathcal{L}$ . We perform a further degeneration of the linear system  $\mathcal{L}_0$  on the central fiber by sending  $\beta$  among the  $n_2$  double points of  $X_2$  to the intersection  $R$  of  $X_1$  and  $X_2$ . As a result we obtain the following systems

$$(3.2) \quad \begin{aligned} \mathcal{L}_1 &:= \mathcal{L}_{(d_1, \dots, d_{r-1}, d_r - k)}(2^{n_1}) & \mathcal{L}_2 &:= \mathcal{L}_{(d_1, \dots, d_{r-1}, k)}(2^{n_2}) \\ \hat{\mathcal{L}}_1 &:= \mathcal{L}_{(d_1, \dots, d_{r-1}, d_r - k - 1)}(2^{n_1}) & \hat{\mathcal{L}}_2 &:= \mathcal{L}_{(d_1, \dots, d_{r-1}, k - 1)}(2^{n_2 - \beta}, 1^\beta), \end{aligned}$$

On the intersection  $R$  we have

$$\mathcal{R}_2 \subseteq \mathcal{L}_{(d_1, \dots, d_{r-1})}(2^\beta).$$

Observe that we are abusing our original notation for  $\mathcal{L}_2$  and  $\hat{\mathcal{L}}_2$  since the  $n_2$  double points on  $X_2$  are no longer in general position. We will say that  $\mathcal{L}_0$  is obtained from  $\mathcal{L}$  by a  $(k, n_2, \beta)$ -degeneration, implying that if  $\beta > 0$  we perform the double degeneration, while if  $\beta = 0$  we do not need to perform it.

#### 4. COMPUTING THE DIMENSION OF THE LIMIT SYSTEM

Our aim is to compute  $\dim(\mathcal{L}_0)$  by recursion. The simplest cases occurs when all the divisors in  $\mathcal{L}_0$  come from a section which is identically zero on one of the two components: in those cases the matching sections of the other system must lie in the kernel of the restriction map.

**Lemma 4.1.** *If  $\mathcal{L}_2$  is empty, then  $\dim(\mathcal{L}_0) = \dim(\hat{\mathcal{L}}_1)$ .*

If, on the contrary, the divisors on  $\mathcal{L}_0$  consist of a divisor on  $X_1$  and a divisor on  $X_2$ , both not identically zero, which match on  $R$ , then the dimension of  $\mathcal{L}_0$  depends on the dimension of the intersection  $\mathcal{R} := \mathcal{R}_1 \cap \mathcal{R}_2$  of the restricted systems.

**Lemma 4.2.**  $\dim(\mathcal{L}_0) = \dim(\mathcal{R}) + \dim(\hat{\mathcal{L}}_1) + \dim(\hat{\mathcal{L}}_2) + 2$ .

*Proof.* A section of  $H^0(X_0, \mathcal{L}_0)$  is obtained by taking an element in  $H^0(R, \mathcal{R})$  and choosing preimages of such an element:  $h^0(X_0, \mathcal{L}_0) = h^0(R, \mathcal{R}) + h^0(X_1, \hat{\mathcal{L}}_1) + h^0(X_2, \hat{\mathcal{L}}_2)$ . Thus, at the linear system level we get the formula.  $\square$

**4.1. Transversality of the restricted systems.** The crucial point is to compute the dimension of  $\mathcal{R}$ . If the systems  $\mathcal{R}_1, \mathcal{R}_2 \subseteq |\mathcal{O}_R(d_1, \dots, d_{r-1})|$  are *transversal*, i.e. if they intersect properly, then the dimension of the intersection  $\mathcal{R}$  is easily computed. It is immediate to see that if  $\hat{\mathcal{L}}_1$  (or  $\hat{\mathcal{L}}_2$ ) is non-special with virtual dimension  $\geq -1$ , and moreover  $\mathcal{L}_1$  (resp.  $\mathcal{L}_2$ ) is non-special, then the restricted system  $\mathcal{R}_1$  (resp.  $\mathcal{R}_2$ ) is the complete linear system  $|\mathcal{O}_R(d_1, \dots, d_{r-1})|$  and transversality trivially holds. If none of the two kernel systems satisfies this property, so that  $\mathcal{R}_1, \mathcal{R}_2$  are both proper subsets of  $|\mathcal{O}_R(d_1, \dots, d_{r-1})|$ , then we need to prove that they intersect properly.

**4.2. Applying the double degeneration technique.** In what follows we will make use of just two types of degenerations, so we consider them in detail now. Choose

$$(4.1) \quad k = r \quad n_2 = \prod_{i=1}^{r-1} (d_i + 1).$$

Then  $v(\mathcal{L}_2) = -1$ . Since transversality trivially holds, and in particular  $\mathcal{R}_1 \cap \mathcal{R}_2 = \emptyset$ , the problem of studying  $\mathcal{L}$  is recursively translated to the problem of studying the lower degree linear system  $\hat{\mathcal{L}}_1 = \mathcal{L}_{(d_1, \dots, d_{r-1}, d_r - r + 1)}(2^{n_1})$  which has the same virtual dimension. This method is simple and useful if one wants to focus on some particular  $r$  and apply induction on the multi-degree  $(d_1, \dots, d_r)$ , provided that the base steps of the induction, i.e. any case  $(d_1, \dots, d_r)$  with  $d_1 \leq \dots \leq d_r \leq r + 1$  are analysed in advance.

However, the aim of this paper is to cover all cases of linear systems of divisors of any multi-degree  $(\mathbb{P}^1)^r$ , for any  $r$ . To this end, we want to exploit induction not only on the multi-degree but also on  $r$  in order to have a more compact and powerful method. This can be done by choosing  $k = 1$  and applying a double degeneration as described in Section 3.3. This argument consists in an ad hoc adaptation of the degeneration technique implied in [Pos10] to prove the non-speciality of linear systems of divisors of degree  $d$  in  $\mathbb{P}^r$  with a general collection of double points.

Choose integers  $k, n_2, \beta$  as follows

$$(4.2) \quad k = 1 \quad \prod_{i=1}^{r-1} (d_i + 1) = r(n_2 - \beta) + \beta, \quad \beta \in \{0, \dots, r - 1\}$$

and perform the double degeneration of  $X, \mathcal{L}$  described above. Observe that elements of  $\hat{\mathcal{L}}_2$  are in bijection with elements of  $\mathcal{L}_{(d_1, \dots, d_{r-1})}(2^{n_2 - \beta}, 1^\beta)$ , that is the linear system of hypersurfaces of  $|\mathcal{O}(d_1, \dots, d_{r-1})|$  on  $(\mathbb{P}^1)^{r-1}$  singular at  $n_2 - \beta$  points and passing through  $\beta$  points, all of them in general position:

$$\hat{\mathcal{L}}_2 = \mathcal{L}_{(d_1, \dots, d_{r-1}, 0)}(2^{n_2 - \beta}, 1^\beta) \cong \mathcal{L}_{(d_1, \dots, d_{r-1})}(2^{n_2 - \beta}, 1^\beta).$$

Observe that the last system has virtual dimension  $-1$  by the definition of  $n_2$  and  $\beta$ . Let  $\pi_r : (\mathbb{P}^1)^r \rightarrow (\mathbb{P}^1)^{r-1}$  be the projection on the first  $r - 1$  factors. Observe that

$$\mathcal{R}_2 \subseteq \mathcal{L}_{(d_1, \dots, d_{r-1})}(1^{n_2 - \beta}, 2^\beta)$$

since elements of  $\mathcal{L}_2$  contain the lines  $\pi_r^{-1}(p_i)$  through each one of the  $n_2$  double points  $p_i$  of  $X_2$ .

**Lemma 4.3.** *Let  $\mathcal{R}_2$  be as above. If  $\mathcal{L}_{(d_1, \dots, d_{r-1})}(2^{n_2})$  is non-special, then  $\mathcal{R}_2 = \mathcal{L}_{(d_1, \dots, d_{r-1})}(1^{n_2 - \beta}, 2^\beta)$ .*

*Proof.* Since  $\mathcal{L}_{(d_1, \dots, d_{r-1})}(2^{n_2})$  is non-special, then  $\mathcal{L}_{(d_1, \dots, d_{r-1})}(2^{n_2}, 1^\beta)$  is non-special due to the fact that the  $\beta$  simple points are in general position on  $R$ . Consider the exact sequence of linear systems:

$$0 \longrightarrow \hat{\mathcal{L}}_2 \longrightarrow \mathcal{L}_2 \longrightarrow \mathcal{R}_2 \longrightarrow 0.$$

Since  $\mathcal{L}_{(d_1, \dots, d_{r-1})}(2^{n_2}, 1^\beta)$  is non-special of virtual dimension  $-1$ , then it is empty. Thus  $\hat{\mathcal{L}}_2$  is empty so that  $\mathcal{R}_2$  is the complete linear system  $\mathcal{L}_{(d_1, \dots, d_{r-1})}(1^{n_2-\beta}, 2^\beta)$  obtained by restricting  $\mathcal{L}_2$  to  $R$ .  $\square$

In order to prove that transversality holds, it is enough to prove that  $\beta$  double points supported on  $R$  impose independent conditions to the linear system  $\mathcal{R}_1$ .

**Lemma 4.4** (Transversality Lemma). *In the same notation as above, if the linear system  $\mathcal{L}_{(d_1, \dots, d_{r-k})}(2^{n_1+\beta})$  is non-special, for points in general position, then  $\mathcal{R}_1$  and  $\mathcal{R}_2$  intersect transversally on  $R$ . In particular*

$$\dim(\mathcal{R}) = \max\{\dim(\mathcal{R}_1) - (n_2 - \beta) - r\beta, -1\}.$$

*Proof.* Since  $\beta < r$ , the scheme formed by  $n_1$  double points on  $X_1$  and  $\beta$  additional double points on  $R$  is general in  $X_1$ . If  $\beta$  nodes in general position impose independent conditions to  $\mathcal{L}_1 = \mathcal{L}_{(d_1, \dots, d_{r-k})}(2^{n_1})$ , namely if  $\mathcal{L}_{(d_1, \dots, d_{r-k})}(2^{n_1+\beta})$  is non-special, then the  $\beta$  nodes supported on  $R$  give independent conditions to  $\mathcal{R}_1$ . Moreover, the intersection  $\mathcal{R}$  is formed by those elements in  $\mathcal{R}_1$  that are singular at  $\beta$  points and pass through  $n_2 - \beta$  points in general position. This proves the formula for  $\dim(\mathcal{R})$ .  $\square$

## 5. THE PROOF OF THE CLASSIFICATION THEOREM

Aim of this section is to prove Theorem 2.1 by induction on the multi-degree and on the dimension  $r$  of the variety.

Define the integers

$$n^- := \left\lfloor \frac{1}{r+1} \prod_{i=1}^r (d_i + 1) \right\rfloor, \quad n^+ := \left\lceil \frac{1}{r+1} \prod_{i=1}^r (d_i + 1) \right\rceil.$$

Notice that if non-speciality holds for a collection of  $n^-$  double points, then it holds for a smaller number of double points. On the other hand, if there are no hypersurfaces of multi-degree  $(d_1, \dots, d_r)$  with  $n^+$  general nodes, the same is true adding other nodes. It is enough to analyse the cases  $n^- \leq n \leq n^+$ .

In what follows we will make use of the following fact proved in [CGG05]:

$$(5.1) \quad \dim \mathcal{L}_{(d_1, \dots, d_r)}(2^n) = \dim \mathcal{L}_d(d - d_1, \dots, d - d_r, 2^n),$$

where  $d = d_1 + \dots + d_r$  and the elements of the right hand side system are hypersurfaces of  $\mathbb{P}^r$  with  $r$  points of multiplicity  $d - d_1, \dots, d - d_r$  and  $n$  double points, all in general position.

**Proposition 5.1.** *If  $d_1, \dots, d_r$  are positive integers as in the following table, then  $\mathcal{L}_{(d_1, \dots, d_r)}(2^n)$  is non-special for any value of  $n$ .*

$r$	$degrees$	$bound$
2	$(d_1, d_2)$	$d_1, d_2 \leq 6$
3	$(d_1, d_2, d_3)$	$d_1, d_2, d_3 \leq 6$
4	$(d_1, d_2, d_3, d_4)$	$d_1, \dots, d_4 \leq 4$
	$(1, 1, d_3, d_4)$	$d_3, d_4 \leq 6$
	$(2, 2, 2, 5)$	
5	$(1, 1, 1, 1, d_5)$	$d_5 \leq 5$

*Proof.* By applying (5.1) we reduce to consider the linear system of  $\mathbb{P}^r$  given by  $\mathcal{L}_d(d - d_1, \dots, d - d_r, 2^n)$ . By means of the programs at this url: <http://www2.udec.cl/~alaface/software/pin.html> we analyze these systems by extracting  $n + r$  random points of  $\mathbb{P}^r$  on the finite field  $\mathbb{F}_{307}$  and calculate the degree  $d$  part of the ideal with the assigned fat points.  $\square$

**5.1. The case  $r = 2$ .** This case is well known in literature, see for example [Laf02, Proposition 5.2] or [VT05]. In any case, for the sake of completeness, we provide a complete proof of this case as well.

**Proposition 5.2.** *Let  $d_1, d_2$  be positive integers. Then  $\mathcal{L}_{(d_1, d_2)}(2^n)$  is special only in the cases described by Proposition 6.1.*

*Proof.* By Proposition 5.1 it is enough to concentrate on the case  $d_3 \geq 6$ . Apply a  $(2, d_1 + 1)$ -degeneration. Then  $v(\mathcal{L}_2) = -1$  and  $v(\hat{\mathcal{L}}_1) = v(\mathcal{L})$ . By induction hypothesis  $\mathcal{L}_2$  is non-special so that it is empty and  $\dim(\mathcal{L}_0) = \dim(\hat{\mathcal{L}}_1)$  by Lemma 4.1. If  $(d_1, d_2)$  is not equal to  $(1, 2a)$ , then  $\hat{\mathcal{L}}_1$  is non-special, by induction so we conclude by Lemma 3.1.  $\square$

**5.2. The case  $r = 3$ .** We will prove the non-speciality of  $\mathcal{L}_{(d_1, d_2, d_3)}(2^n)$ , with all the  $d_i$  positive and  $(d_1, d_2, d_3, n) \neq (2, 2, 2, 7), (1, 1, 2a, 2a + 1)$ . This case also was previously known, see [CGG07] or [BD10].

**Proposition 5.3.** *Let  $d_1, d_2, d_3$  be positive integers. Then  $\mathcal{L}_{(d_1, d_2, d_3)}(2^n)$  is special only in the cases described by Proposition 6.1.*

*Proof.* By Proposition 5.1 it is enough to concentrate on the case  $d_3 \geq 6$ . Apply a  $(3, n_2)$ -degeneration with  $n_2 := (d_1 + 1)(d_2 + 1)$ . Then  $v(\mathcal{L}_2) = -1$  and  $v(\hat{\mathcal{L}}_1) = v(\mathcal{L})$ . Observe that by induction hypothesis  $\mathcal{L}_2$  is non-special so that it is empty and  $\dim(\mathcal{L}_0) = \dim(\hat{\mathcal{L}}_1)$  by Lemma 4.1. If  $(d_1, d_2, d_3)$  is not equal to  $(1, 1, 2a)$ , then  $\hat{\mathcal{L}}_1$  is non-special, by induction so we conclude by Lemma 3.1.  $\square$

**5.3. The case  $r = 4$ .** We proceed with our investigation by proving the non-speciality of  $\mathcal{L}_{(d_1, \dots, d_4)}(2^n)$  for  $(d_1, d_2, d_3, d_4)$  distinct from  $(1, 1, 1, 1)$ .

**Proposition 5.4.** *Let  $d_1, d_2, d_3, d_4$  be positive integers. Then  $\mathcal{L}_{(d_1, \dots, d_4)}(2^n)$  is special only in the cases described by Proposition 6.1.*

*Proof.* We begin by analyzing four distinct cases.

- (1)  $(d_1, d_2, d_3, d_4) = (1, 1, 1, d_4)$  with  $d_4 \geq 7$ . In this case we perform a  $(4, 8)$ -degeneration obtaining the systems

$$\mathcal{L}_1 = \mathcal{L}_{(1, 1, 1, d_4 - 4)}(2^{n-8}) \quad \mathcal{L}_2 = \mathcal{L}_{(1, 1, 1, 4)}(2^8).$$

Since  $\mathcal{L}_2$  is empty by Proposition 5.1 and  $\hat{\mathcal{L}}_1 = \mathcal{L}_{(1, 1, 1, d_4 - 5)}(2^{n-8})$  is non-special by induction on  $d_4$ , we conclude using Lemma 4.1.



- (2)  $(d_1, d_2, d_3, d_4) = (1, 1, 4, d_4)$ , with  $d_4 \geq 6$ . In this case we perform a  $(4, 20)$ -degeneration obtaining the systems

$$\mathcal{L}_1 = \mathcal{L}_{(1,1,4,d_4-4)}(2^{n-20}) \quad \mathcal{L}_2 = \mathcal{L}_{(1,1,4,4)}(2^{20}).$$

Since  $\mathcal{L}_2$  is empty by Proposition 5.1 and  $\hat{\mathcal{L}}_1 = \mathcal{L}_{(1,1,4,d_4-5)}(2^{n-20})$  is non-special by induction on  $d_4$ , we conclude using Lemma 4.1.

- (3)  $(d_1, d_2, d_3, d_4) = (1, 1, d_3, d_4)$ , with  $d_3, d_4 \geq 6$ . In this case we perform a  $(4, 4d_4 + 4)$ -degeneration obtaining the systems

$$\mathcal{L}_1 = \mathcal{L}_{(1,1,d_3,d_4-4)}(2^{n-4d_4-4}) \quad \mathcal{L}_2 = \mathcal{L}_{(1,1,d_3,4)}(2^{4d_4+4}).$$

Since  $\mathcal{L}_2$  is empty by Proposition 5.1 and  $\hat{\mathcal{L}}_1 = \mathcal{L}_{(1,1,d_3,d_4-5)}(2^{n-4d_4-4})$  is non-special by induction on  $d_4$ , we conclude using Lemma 4.1.

- (4)  $(d_1, d_2, d_3, d_4) = (2, 2, 2, d_4)$ , with  $d_4 \geq 6$ . In this case we perform a  $(4, 27)$ -degeneration obtaining the systems

$$\mathcal{L}_1 = \mathcal{L}_{(2,2,2,d_4-4)}(2^{n-27}) \quad \mathcal{L}_2 = \mathcal{L}_{(2,2,2,4)}(2^{27}).$$

Since  $\mathcal{L}_2$  is empty by Proposition 5.1 and  $\hat{\mathcal{L}}_1 = \mathcal{L}_{(2,2,2,d_4-5)}(2^{n-27})$  is non-special by induction on  $d_4$ , we conclude using Lemma 4.1.

Suppose now that  $(d_1, d_2, d_3, d_4)$  is distinct from  $(1, 1, 1, d_4)$ ,  $(2, 2, 2, d_4)$  and  $(1, 1, 2a, d_4)$ . We perform a  $(1, n_2, \beta)$ -degeneration, with  $n_2$  and  $\beta$  defined as in (4.2) obtaining the systems

$$\mathcal{L}_1 = \mathcal{L}_{(d_1,d_2,d_3,d_4-1)}(2^{n_1}) \quad \mathcal{L}_2 = \mathcal{L}_{(d_1,d_2,d_3,1)}(2^{n_2}).$$

Recall that exactly  $\beta$  of the  $n_2$  double points of  $X_2$  are sent to  $R$ . Thus the kernels are:

$$\hat{\mathcal{L}}_1 = \mathcal{L}_{(d_1,d_2,d_3,d_4-2)}(2^{n_1}) \quad \hat{\mathcal{L}}_2 = \mathcal{L}_{(d_1,d_2,d_3,0)}(2^{n_2-\beta}, 1^\beta).$$

Observe that  $\hat{\mathcal{L}}_2 \cong \mathcal{L}_{(d_1,d_2,d_3)}(2^{n_2-\beta}, 1^\beta)$  and the last system is empty by induction since it has virtual dimension  $-1$ . Also  $\hat{\mathcal{L}}_1$  is empty since it is non-special by induction and has negative virtual dimension. Thus  $\mathcal{L}$  is non-special by the assumption on the  $d_i$  and by Lemma 4.2 and Lemma 4.4.  $\square$

**5.4. The case  $r = 5$ .** We show that there are no special systems in dimension 5.

**Proposition 5.5.** *Let  $d_1, \dots, d_5$  be positive integers. Then  $\mathcal{L}_{(d_1, \dots, d_5)}(2^n)$  is non-special.*

*Proof.* If  $(d_1, d_2, d_3, d_4, d_5) = (1, 1, 1, 1, d_5)$ , with  $d_5 \geq 6$ , we perform a  $(5, 16)$ -degeneration obtaining the linear systems

$$\mathcal{L}_1 = \mathcal{L}_{(1,1,1,1,d_5-5)}(2^{n-16}) \quad \mathcal{L}_2 = \mathcal{L}_{(1,1,1,1,5)}(2^{16}).$$

Since  $\mathcal{L}_2$  is empty by Proposition 5.1 and  $\hat{\mathcal{L}}_1 = \mathcal{L}_{(1,1,1,1,d_5-6)}(2^{n-16})$  is non-special by induction, we conclude using Lemma 4.1.

Suppose now  $(d_1, d_2, d_3, d_4, d_5) \neq (1, 1, 1, 1, d_5)$ . Like in the four dimensional case we perform a  $(1, n_2, \beta)$ -degeneration, with  $n_2$  and  $\beta$  defined as in (4.2) obtaining the systems

$$\mathcal{L}_1 = \mathcal{L}_{(d_1, \dots, d_4, d_5-1)}(2^{n_1}) \quad \mathcal{L}_2 = \mathcal{L}_{(d_1, \dots, d_4, 1)}(2^{n_2}).$$

Recall that exactly  $\beta$  of the  $n_2$  double points of  $X_2$  are sent to  $R$ . Thus the kernels are:

$$\hat{\mathcal{L}}_1 = \mathcal{L}_{(d_1, \dots, d_4, d_5-2)}(2^{n_1}) \quad \hat{\mathcal{L}}_2 = \mathcal{L}_{(d_1, \dots, d_4, 0)}(2^{n_2-\beta}, 1^\beta).$$

Observe that  $\hat{\mathcal{L}}_2 \cong \mathcal{L}_{(d_1, d_2, d_3, d_4)}(2^{n_2-\beta}, 1^\beta)$  and the last system is empty by induction since it has virtual dimension  $-1$ . Also  $\hat{\mathcal{L}}_1$  is empty since it is non-special by induction and has negative virtual dimension. Thus  $\mathcal{L}$  is non-special by the assumption on the  $d_i$  and by Lemma 4.2.  $\square$

**5.5. The cases  $r \geq 6$ .** We show that there are no special systems in dimension  $\geq 6$ .

**Proposition 5.6.** *Let  $r \geq 6$ . Let  $d_1, \dots, d_r$  be positive integers. Then the linear system  $\mathcal{L}_{(d_1, \dots, d_r)}(2^n)$  is non-special.*

*Proof.* We proceed by induction on  $d := d_1 + \dots + d_r$ . If  $d = r$ , then  $d_i = 1$  for any  $i$ . This case is non-special as proved in [CGG11]. If  $d > r$ , then after possibly reordering the  $d_i$  we can assume that  $d_r > 1$ . We perform a  $(1, n_2, \beta)$ -degeneration, with  $n_2$  and  $\beta$  defined as in (4.2) obtaining the systems

$$\mathcal{L}_1 = \mathcal{L}_{(d_1, \dots, d_{r-1}, d_r-1)}(2^{n_1}) \quad \mathcal{L}_2 = \mathcal{L}_{(d_1, \dots, d_{r-1}, 1)}(2^{n_2}).$$

Recall that exactly  $\beta$  of the  $n_2$  double points of  $X_2$  are sent to  $R$ . Thus the kernels are:

$$\hat{\mathcal{L}}_1 = \mathcal{L}_{(d_1, \dots, d_{r-1}, d_r-2)}(2^{n_1}) \quad \hat{\mathcal{L}}_2 = \mathcal{L}_{(d_1, \dots, d_{r-1}, 0)}(2^{n_2-\beta}, 1^\beta).$$

Observe that  $\hat{\mathcal{L}}_2 \cong \mathcal{L}_{(d_1, \dots, d_{r-1})}(2^{n_2-\beta}, 1^\beta)$  and the last system is empty by induction since it has virtual dimension  $-1$ . The kernel system  $\hat{\mathcal{L}}_1$  has negative virtual dimension for any  $(d_1, \dots, d_r)$ , as one can easily check. Furthermore the dimension of the limit system equals the dimension of the intersection  $\mathcal{R}_1 \cap \mathcal{R}_2$  by Lemma 4.2. Indeed, since  $\mathcal{R}_1$  and  $\mathcal{R}_2$  intersect transversally by Lemma 4.4, we have

$$\begin{aligned} \dim(\mathcal{R}_1 \cap \mathcal{R}_2) &= \max\{\dim(\mathcal{R}_1) - (n_2 - \beta) - r\beta, -1\} \\ &= \max\left\{\prod_{i=1}^r (d_i + 1) - 1 - (r+1)n, -1\right\} = e(\mathcal{L}). \end{aligned}$$

$\square$

## 6. SPECIAL LINEAR SYSTEMS

In this section we complete the proof of Theorem 2.1 by calculating the dimension of each special system.

**Proposition 6.1.** *The following linear systems have the stated virtual and effective dimensions. In each case the system  $\mathcal{L}$  is singular along a smooth rational variety.*

$r$	degrees	$n$	$v(\mathcal{L})$	$\dim(\mathcal{L})$
2	(2, 2a)	$2a + 1$	-1	0
3	(1, 1, 2a)	$2a + 1$	-1	0
	(2, 2, 2)	7	-2	0
4	(1, 1, 1, 1)	3	0	1

*Proof.* Observe that Proposition 5.1 already provide a computer based proof of the first part of the statement. Anyway we prefer to give an alternative proof also for the calculation of the dimension in order to make things more explicit. The dimension of each such system can be found by repeated use of formula (5.1) together with the following one (see [LU10]). Let  $\mathcal{L}^{\mathbb{P}^n} := \mathcal{L}_d(m_1, \dots, m_{n+1}, m_{n+2}, \dots, m_r)$ , then

$$(6.1) \quad \dim \mathcal{L}^{\mathbb{P}^n} = \dim \mathcal{L}_{d+k}(m_1 + k, \dots, m_{n+1} + k, m_{n+2}, \dots, m_r),$$

where  $k = (n-1)d - m_1 - \dots - m_{n+1}$ . Our analysis begins with the two dimensional case. If  $r = 2$  by applying (5.1) first and (6.1) after we get

$$\begin{aligned} \dim \mathcal{L}_{(2,2a)}(2^{2a+1}) &= \dim \mathcal{L}_{2a+2}(2a, 2^{2a+2}) \\ &= \dim \mathcal{L}_{2a}(2(a-1), 2^{2a}) \\ &= \dim \mathcal{L}_2(2^2) = 0. \end{aligned}$$

The linear system is singular along the unique smooth rational curve in  $\mathcal{L}_{(1,a)}(1^{2a+1})$ . If  $r = 3$  we have to analyze two cases. The first one is

$$\begin{aligned} \dim \mathcal{L}_{(1,1,2a)}(2^{2a+1}) &= \dim \mathcal{L}_{2a+2}((2a+1)^2, 2^{2a+2}) \\ &= \dim \mathcal{L}_{2a}(2a-1, 2^{2a}) \\ &= \dim \mathcal{L}_2(2^2, 1^2) = 0. \end{aligned}$$

Observe that the dimension of the last system is 0 since it contains just a pair of planes which intersect along a line through the first two points. These two planes can be seen also in the original system. They are  $D_1 + D_2 \in \mathcal{L}_{(1,1,2a)}(2^{2a+1})$ , with  $D_1 \in \mathcal{L}_{(1,0,a)}(1^{2a+1})$  and  $D_2 \in \mathcal{L}_{(0,1,a)}(1^{2a+1})$ . Thus the system is singular along the curve  $D_1 \cap D_2$  which is smooth by Bertini's theorem and rational by the adjunction formula. The second case in dimension  $r = 3$  is

$$\begin{aligned} \dim \mathcal{L}_{(2,2,2)}(2^7) &= \dim \mathcal{L}_6(4^3, 2^7) \\ &= \dim \mathcal{L}_4(2^9) = 0. \end{aligned}$$

This last system is well known from Alexander-Hirschowitz theorem, see [AH95, Pos10]. Observe that  $\mathcal{L}_{(2,2,2)}(2^7)$  is singular along the smooth rational surface defined by  $\mathcal{L}_{(1,1,1)}(1^7)$ . Our last case is when  $r = 4$ . Here we have

$$\begin{aligned} \dim \mathcal{L}_{(1,1,1,1)}(2^3) &= \dim \mathcal{L}_4(3^4, 2^3) \\ &= \dim \mathcal{L}_2(2^2, 1^4) \\ &= \dim \mathcal{L}_1(1^3) = 1. \end{aligned}$$

Consider now the linear system  $\mathcal{L}_{(1,1,1,1)}(2^3)$ , where the three double points have coordinates:  $p_1 := ([1 : 0], [1 : 0], [1 : 0], [1 : 0])$ ,  $p_2 := ([0 : 1], [0 : 1], [0 : 1], [0 : 1])$ ,  $p_3 := ([1 : 1], [1 : 1], [1 : 1], [1 : 1])$ . An easy calculation shows that the elements of the linear system are zero locus of the sections of the pencil:

$$\alpha_0 (x_0 y_1 - x_1 y_0)(z_0 w_1 - z_1 w_0) + \alpha_1 (x_0 z_1 - x_1 z_0)(y_0 w_1 - y_1 w_0).$$

In particular the general element of the pencil is singular along the curve  $V(x_0 y_1 - x_1 y_0) \cap V(z_0 w_1 - z_1 w_0) \cap V(x_0 z_1 - x_1 z_0) \cap V(y_0 w_1 - y_1 w_0)$ . This is exactly the image of the diagonal morphism  $\Delta : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ , hence it is a smooth rational curve.  $\square$

**6.1. The 7-secant variety of the  $(2, 2, 2)$ -embedding of  $(\mathbb{P}^1)^3$ .** Our list of special systems contains the case  $\mathcal{L}_{(2,2,2)}(2^7)$  which has virtual dimension  $-2$  and dimension 0. In this section we will show that the corresponding secant variety enjoys a symmetry which allows us to determine its equation.

Let  $\varphi : (\mathbb{P}^1)^3 \rightarrow \mathbb{P}^{26}$  be the Segre-Veronese embedding of  $(\mathbb{P}^1)^3$  defined by the complete linear system  $\mathcal{L}_{(2,2,2)}$ , and let  $S$  be the 7-secant variety to the image of  $\varphi$ .

Denote, by abuse of notation, with the same letter the corresponding linear map of vector spaces

$$\begin{aligned}\varphi : V \otimes V \otimes V &\longrightarrow \operatorname{Sym}^2(V) \otimes \operatorname{Sym}^2(V) \otimes \operatorname{Sym}^2(V) \\ a \otimes b \otimes c &\longmapsto a^2 \otimes b^2 \otimes c^2,\end{aligned}$$

where each copy of  $V$  represents the vector space of degree 1 homogeneous polynomials in two variables. Let  $V^*$  be the dual of  $V$ . Let  $W := \operatorname{Sym}^2(V)^{\otimes 3}$  be the codomain of  $\varphi$ . Given a polynomial  $f \in W$ , it belongs to the 7-secant variety  $S$  if and only if

$$f = a_1^2 b_1^2 c_1^2 + \cdots + a_7^2 b_7^2 c_7^2,$$

where we are adopting the short notation  $abc$  for  $a \otimes b \otimes c$ . Consider the catalecticant map

$$C_f : (V^*)^{\otimes 3} \rightarrow V^{\otimes 3} \quad a^* b^* c^* \mapsto a^* b^* c^*(f).$$

We have that if  $f \in S$ , then  $\operatorname{rk}(C_f) \leq 7$ . Hence, as  $f$  varies in  $W$ , the determinant of  $C_f$  gives an equation that vanishes on  $S$ . Since this equation turns out to be irreducible and  $S$  is an irreducible hypersurface, this will provide us the equation of  $S$ . In coordinates, let  $a_i b_j c_k$ , with  $i, j, k \in \{0, 1\}$  be a basis of  $V^{\otimes 3}$  and let  $a_i^* b_j^* c_k^*$  be a basis of the dual vector space  $(V^*)^{\otimes 3}$  such that  $a_i^* b_j^* c_k^*(a_i b_j c_k) = 1$ . An element  $f \in W$  can be uniquely written as

$$f = z_0 a_0^2 b_0^2 c_0^2 + z_1 a_0 a_1 b_0^2 c_0^2 + \cdots + z_{26} a_1^2 b_1^2 c_1^2.$$

We have that, for example,  $a_0^* b_0^* c_0^*(f) = 8z_0 a_0 b_0 c_0 + 4z_1 a_1 b_0 c_0 + 4z_3 a_0 b_1 c_0 + 2z_4 a_1 b_1 c_0 + 4z_9 a_0 b_0 c_1 + 2z_{10} a_1 b_0 c_1 + 2z_{12} a_0 b_1 c_1 + z_{13} a_1 b_1 c_1$ . Thus the equation of  $S$  is given by the vanishing of the determinant

$$\begin{vmatrix} 8z_0 & 4z_9 & 4z_3 & 2z_{12} & 4z_1 & 2z_{10} & 2z_4 & z_{13} \\ 4z_1 & 2z_{10} & 2z_4 & z_{13} & 8z_2 & 4z_{11} & 4z_5 & 2z_{14} \\ 4z_3 & 2z_{12} & 8z_6 & 4z_{15} & 2z_4 & z_{13} & 4z_7 & 2z_{16} \\ 2z_4 & z_{13} & 4z_7 & 2z_{16} & 4z_5 & 2z_{14} & 8z_8 & 4z_{17} \\ 4z_9 & 8z_{18} & 2z_{12} & 4z_{21} & 2z_{10} & 4z_{19} & z_{13} & 2z_{22} \\ 2z_{10} & 4z_{19} & z_{13} & 2z_{22} & 4z_{11} & 8z_{20} & 2z_{14} & 4z_{23} \\ 2z_{12} & 4z_{21} & 4z_{15} & 8z_{24} & z_{13} & 2z_{22} & 2z_{16} & 4z_{25} \\ z_{13} & 2z_{22} & 2z_{16} & 4z_{25} & 2z_{14} & 4z_{23} & 4z_{17} & 8z_{26} \end{vmatrix} = 0.$$

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